HW5 Solution.

1 $\quad X(t)=\int_0^t x'(s)ds + X(0)$

$= \int_0^t (e^s, 8s^2+2)ds + (1,3)$

$= (e^t-1, \frac{8}{3}t^3+2t)+(1,3)$

$= (e^t, \frac{8}{3}t^3+2t+3)$

2. (1) $\quad x'(t)-x(t)=2t$

multiply both sides by e^{-t}

$(e^{-t}x(t))' = 2te^{-t}$

$e^{-t}x(t) = \int 2te^{-t}dt = -2te^{-t}-2e^{-t}+C_1$

$x(t) = -2t-2+C_1e^t$

$x(0)=-2+C_1=2 \Rightarrow C_1=4$

$x(t)=-2t-2+4e^t$

(2) $\quad y'(t)=y(t)-t^2$

multiply both sides by e^{-t}

$(e^{-t}y(t))' = -t^2e^{-t}$

$e^{-t}y(t) = -e^{-t}(-2-2t-t^2)+C_2$

$y(t) = 2+2t+t^2+C_2e^t$

$y(0)=2+C_2=-1 \quad C_2=-3$

$y(t)=2+2t+t^2-3e^t$

3. $\det(\lambda I - A) = \det\begin{pmatrix} \lambda & -2 \\ -2 & \lambda \end{pmatrix} = \lambda^2 - 4$

$\lambda_1 = 2 \quad \lambda_2 = -2$

$(\lambda_1 I - A)\vec{v_1} = 0 \qquad \vec{v_1} = (1,1)$

$(\lambda_2 I - A)\vec{v_2} = 0 \qquad \vec{v_2} = (1,-1)$

Let $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad M^{-1}AM = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

$M^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

$(M^{-1}X(t))' = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} M^{-1}X(t) + M^{-1}\begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$

Let $\quad Y(t) = M^{-1}X(t)$

The equation becomes

$Y'(t) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} Y(t) + \begin{pmatrix} \frac{t}{2} + \frac{1}{2} \\ \frac{t}{2} - \frac{1}{2} \end{pmatrix}$

$Y(0) = (0,1)$

$Y(t) = \left(-\frac{3}{8} - \frac{t}{4} + \frac{3}{8}e^{2t},\ -\frac{3}{8} + \frac{t}{4} + \frac{11}{8}e^{-2t}\right)$

$X(t) = MY(t) = \left(-\frac{3}{4} + \frac{3}{8}e^{2t} + \frac{11}{8}e^{-2t},\ -\frac{t}{2} + \frac{3}{8}e^{2t} - \frac{11}{8}e^{-2t}\right)$